MATH 20D Spring 2023 Lecture 19.

The Laplace Transforms Method for Solving IVP's and Transforms of Discontinuous Functions.

Outline

- Midterm 2 is next Wednesday during lecture.
- Midterm 2 is a cumulative exam for the material covered in lectures 1-20 and homeworks 1-6. However the emphasis of the exam will be the topics covered in
  - Homeworks 4, 5, and 6.
  - Lectures 11-20.
- Students seeking additional review problems are advised to study the exercise sets from Nagle, Saff, and Snider textbook sections 4.6, 4.7, 7.2, 7.3, 7.4, 7.6, & 7.9.
- Students are permitted the use of one double sided page of handwritten notes together with **scientific** calculator during the exam. No other electronic devices are permitted during the exam.

Contents

- Recall if $f'(t)$ is differentiable and of exponential order α then

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Differential Equation in t-space $\longrightarrow$ Algebraic Equation in s-space

Example

(a) Using the method of Laplace transform, solve the initial value problem

$$y'' + 9y = 0; \qquad y(0) = 3, \quad y'(0) = 0.$$

Example

Solve the initial value problem

$$y'' - 2y' + 5y = -8e^{-t}, \qquad y(0) = 2, \quad y'(0) = 12.$$

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Write a differential equation governing the amount of salt in the tank at time t.

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Theorem

Suppose $F(s) = \mathscr{L}\{f(t)\}(s)$ *exists for* $s > \alpha \geqslant 0$. *If* a *is a positive constant, then*

$$\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s),$$

and conversely $\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)$.

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(d) $\mathscr{L}\{\Pi_{a,b}(t)\}(s)$ with $0 \leqslant a < b < \infty$ constant and

$$\Pi_{a,b}(t) = \begin{cases} 0, & t < a \text{ or } t > b, \\ 1, & a < t < b. \end{cases}$$

Example

Using the method of Laplace transform, solve the initial value problem

$$y''(t) + 4y(t) = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 < t < 1, \\ -1, & 1 < t < 2, \\ 0, & 2 < t. \end{cases}$$